Math 246C Lecture 11 Notes

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1 Weyl's Lemma and Perron's Method

1.1 Weyl's lemma

Last time, we were talking about Green's functions for $\Omega \subseteq \mathbb{C}$:

$$G(x,y) = \frac{1}{2\pi} \log|x-y| + h_x(y), \qquad G(x,y) = 0, y \in \partial\Omega,$$

where h_x is harmonic. If

$$E(x) = \frac{1}{2\pi} \log|x|,$$

then E is a fundamental solution of Δ: for all $\varphi \in C_0^\infty(\mathbb{R}^2)$:

$$\int E \Delta \varphi = \varphi(0).$$

Theorem 1.1 (Weyl's lemma)**.** *Let $\Omega \subseteq \mathbb{C}$ be open, and let $u \in L^1_{\mathrm{loc}}(\Omega)$ be such that*

$$\int u \Delta \varphi \, dx = 0 \qquad \forall \in C_0^\infty(\Omega).$$

Then there exists a harmonic $u_1 \in C^\infty(\Omega)$ such that $u = u_1$ a.e. in Ω.

Proof. Let $\omega \subseteq \Omega$ be open with compact $\overline{\omega} \subseteq \Omega$, and let $\psi \in C_0^\infty$ with $\psi = 1$ near $\overline{\omega}$. Let

$$w(x,y) = \Delta_y((1-\psi(y))E(x-y)), \qquad x \in \omega, y \in \Omega.$$

Then $w \in C^\infty$, and $y \mapsto w(x,y)$ has compact support: for all $x \in \omega$,

$$w(x,y) = (1-\psi(y)) \underbrace{(\Delta E)(x-y)}_{=0} + \underbrace{\cdots}_{\text{has supp} \subseteq \operatorname{supp}(\nabla\psi) \subseteq \Omega}.$$

Let $v(x) = \int u(y)w(x,y)\,dy \in C^\infty(\omega)$. We claim that for all $g \in C_0^\infty(\omega)$, the integral $\int v(x)g(x)\,dx = \int u(x)g(x)\,dx$; this implies that $u = v$ a.e. We have:

$$\begin{aligned}
\int v(x)g(x)\,dx &= \iint u(y)\Delta_y((1-\psi(y))E(x-y))g(x)\,dx\,dy \\
&= \int u(y)\Delta_y \left[(1-\psi(y)) \underbrace{\int E(x-y)g(x)\,dx}_{h(y)} \right] dy \\
&= \int u(y)\Delta_y((1-\psi(y))h(y))\,dy
\end{aligned}$$

Here, $h(y) = \int E(x)g(x+y)\,dx \in C^\infty(\mathbb{R}^2)$, where $E \in L^1_{\text{loc}}$, $\psi h \in C_0^\infty(\Omega)$.

$$= \int u(y)\Delta h(y)\,dy - \underbrace{\int u(y)\Delta(\psi h)\,dy}_{=0}$$

E is a fundamental solution to the Lapalacian, so $\Delta h(y) = \int E(x)\Delta g(x+y)\,dx = g(y)$.

$$= \int u(y)y(y)\,dy. \qquad \square$$

Remark 1.1. The argument in the proof only uses that $E \in L^1_{\text{loc}}$ and $E \in C^\infty(\mathbb{R}^2 \setminus \{0\})$. If we replaced the Laplacian by any other operator with a fundamental solution, the same proof would work.

1.2 Perron's method for constructing harmonic functions

Recall Perron's method for $\Omega \subseteq \mathbb{C}$:

Lemma 1.1. *Let $\Omega \subseteq \mathbb{C}$ be open and connected, and let $u : \Omega \to [-\infty, \infty)$ be subharmonic with $u \not\equiv -\infty$. Let $D = \{|x-a| < R\}$ be such that $\overline{D} \subseteq \Omega$, and define*

$$u_D(x) = \begin{cases} u(x) & x \in \Omega \setminus D \\ \frac{1}{2\pi R}\int_{|y|=R} P_R(x-a,y)u(a+y)\,ds(y) & x \in D. \end{cases}$$

Then u_D is subharmonic in Ω, and $u \le u_D$.

The function u_D is called the **Poisson modification** of u.

Definition 1.1. Let $\Omega \subseteq \mathbb{C}$ be open and connected. A **continuous Perron family** in Ω is a family $\mathcal{F}$ of continuous subharmonic functions $u : \Omega \to [-\infty, \infty)$ such that

1. $u, v \in \mathcal{F} \implies \max(u, v) \in \mathcal{F}$.

2. If $u \in \mathcal{F}$ and D is a disc with $\overline{D} \subseteq \Omega$, then $u_D \in \mathcal{F}$.

3. For each $x \in \Omega$, there is a $u \in \mathcal{F}$ such that $u(x) > -\infty$.

Theorem 1.2 (Perron's method)**.** *Let $\mathcal{F}$ be a continuous Perron family on an open and connected $\Omega \subseteq \mathbb{C}$, and let $u = \sup_{v \in \mathcal{F}} v$ pointwise. Then one of the following statements holds:*

1. $u(x) \equiv +\infty$ for all $x \in \Omega$.

2. u is harmonic in Ω.

Remark 1.2. The proof is of local nature; it uses only local properties if $v \in \mathcal{F}$, and the maximum principle is only used on small discs in Ω.

Let X be a Riemann surface. We claim that Perron's method works on X.

Definition 1.2. A function $u : X \to [-\infty, \infty)$ is **subharmonic** (resp. **harmonic**) if for every complex chart $\varphi_\alpha : U_\alpha \to V_\alpha$ in some atlas, $u \circ \varphi_\alpha^{-1}$ is subharmonic (resp. **harmonic**) in V_α.

Definition 1.3. A **parametric disc** $D = D_X \subseteq X$ is a set such that there exists a complex chart $\varphi : U \to V$ such that $\overline{D}_X \subseteq U$ and $\varphi(D_X)$ is a Euclidean disc.

Given $u \in SH(X)$, define its **Poisson modification**:

$$u_{D_X}(x) = \begin{cases} u(x) & x \in X \setminus D \\ h(x) & x \in D, \end{cases}$$

where h is a harmonic extension of $u|_{\partial D}$.

The fundamental theorem of Perron's method is valid on X, so we can construct integrable harmonic functions on X.